STSB6816 Test 2 of 2023

Mathematical Statistics and Actuarial Science; University of the Free State

2023/05/18

## Time: 180 minutes; Marks: 50

# MEMORANDUM

# Instructions

* Answer all questions in a single R Markdown document. Please knit to PDF or Word at the end and submit both the PDF/Word document and the “.Rmd” file for assessment, in that order.
* Label questions clearly, as it is done on this question paper.
* All results accurate to about 3 decimal places.
* Show all derivations, formulas, code, sources, and reasoning.
* Intervals should cover 95% probability unless stated otherwise.
* No communication software, devices, or communication capable websites may be accessed prior to submission. You may not (nor even appear to) attempt to communicate or pass information to another student.

# Introduction

The data is provided at <https://ufs.blackboard.com>. **It consists of the following columns: Response\_ID, Respondent\_Name, Respondent\_ID, Year, Response\_Text, Response\_Numeric.**

A pastor is tracking their congregation’s views on a particular matter. To do this they set up a survey where they ask whether people agree with a statement on that viewpoint using a 7 point Likert scale (Strongly Disagree, …, Strong Agree). The survey is sent out year after year for a few years and then the time comes to analyse it. The pastor has some concerns regarding the data and needs your help.

* It seems that mostly the responses come from the same people year after year (mostly the choir).
* They can’t decide whether to model the responses as ordinal data using a categorical distribution, interval data using binomial distribution, or numeric using a normal distribution.

He asks you to build and compare these three as mixed effects models, each with Year as a linear fixed effect on the underlying scale and Respondent as a random intercept effect. Then he wants you to use the best model to determine whether the people are agreeing more with the statement.

# Question 1

**1.1)** Explain why Respondent should be included in the model as an effect **at all**. **[3]**

**1.2)** Explain why Respondent should be included in the model as **random** effect. **[2]**

**1.3)** Explain what including Respondent only as an intercept term implies with regard to the assumed slopes of each congregant over time. **[2]**

**1.4)** Import the data set into R and explore it visually. You could use a box plot with Year on the x axis perhaps. Discuss what you see. **[4]**

"STSB6816Test2Data2023.xlsx" |> openxlsx::read.xlsx("TestData") -> d

library(tidyverse)

data.frame(Year = d$Year, y = d$Response\_Numeric, s = d$Respondent\_Name) |>
 ggplot(aes(x = Year, y = y)) +
 geom\_boxplot(aes(group = Year)) +
 geom\_smooth(method = 'lm', formula = 'y~x') +
 geom\_jitter(aes(colour = s), width = 0.2, height = 0.2)



**1.5)** Fit a standard mixed effects model assuming that the numerically encoded responses follow a conditional normal distribution given the year number as a continuous linear predictor and respondent as a random intercept. Summarise the distribution of the coefficient of the year number. **[8]**

library(rstan)
mycores <- 3
options(mc.cores = mycores)

data {
 int n;
 vector[n] y;
 vector[n] x;
 int n\_s;
 int subj\_ind[n];
}
parameters {
 real beta0;
 real beta1;
 real<lower=0> sigma;
 real z[n\_s];
 real<lower=0> tau;
}
transformed parameters {
 vector[n] mu;
 for (i in 1:n) {
 mu[i] = beta0 + beta1\*x[i] + z[subj\_ind[i]];
 }
}
model {
 y ~ normal(mu, sigma);
 z ~ normal(0, tau);
 target += -2\*log(sigma) - 2\*log(tau);
}
generated quantities {
 vector[n] log\_lik;
 for (i in 1:n) {
 log\_lik[i] = normal\_lpdf(y[i] | mu[i], sigma);
 }
}

saveRDS(SimpleRE, file = 'SimpleRE.Rds')

Model1Fit <- sampling(SimpleRE,
 list(n = nrow(d),
 x = d$Year,
 y = d$Response\_Numeric,
 n\_s = max(d$Respondent\_ID),
 subj\_ind = d$Respondent\_ID),
 iter = 10000,
 chains = mycores)

| Warning: There were 2 divergent transitions after warmup. See
| https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
| to find out why this is a problem and how to eliminate them.

| Warning: There were 3107 transitions after warmup that exceeded the maximum treedepth. Increase max\_treedepth above 10. See
| https://mc-stan.org/misc/warnings.html#maximum-treedepth-exceeded

| Warning: Examine the pairs() plot to diagnose sampling problems

| Warning: The largest R-hat is 1.73, indicating chains have not mixed.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#r-hat

| Warning: Bulk Effective Samples Size (ESS) is too low, indicating posterior means and medians may be unreliable.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#bulk-ess

| Warning: Tail Effective Samples Size (ESS) is too low, indicating posterior variances and tail quantiles may be unreliable.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#tail-ess

pars\_of\_interest <- c('beta1')
Model1Fit |> traceplot(pars = pars\_of\_interest)



summary(Model1Fit, pars = pars\_of\_interest)$summary |> kable(digits = 3)

|  | mean | se\_mean | sd | 2.5% | 25% | 50% | 75% | 97.5% | n\_eff | Rhat |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| beta1 | 0.261 | 0.033 | 0.09 | 0.06 | 0.204 | 0.288 | 0.318 | 0.413 | 7.537 | 1.112 |

**1.6)** According to the above model, what is the probability that the agreement with the viewpoint is increasing by over 0.05 steps per year (i.e., $P\left[β\_{1}>0.05\right]$)? **[3]**

Model1Sims <- rstan::extract(Model1Fit)
cat('\n\n$P[\\beta\_1>0.05]=$', round(mean(Model1Sims$beta1 > 0.05),3), '\n\n')

$P\left[β\_{1}>0.05\right]=$ 0.979

**1.7)** Adapt the mixed effects model to assume that the numerically encoded responses follow a conditional **binomial** distribution given the year number as a continuous linear predictor and respondent as a random intercept on the logistic scale. Fit the model and summarise the distribution of the coefficient of the year number. **[5]**

Hint: The likelihood component of the model can be expressed mathematically as

$$\begin{matrix}y\_{i}&∼binomial\left(6,π\_{i}\right) i=1…n\\π\_{i}&=logit^{−1}\left(β\_{0}+β\_{1}\*x\_{i}+z\_{r\_{i}}\right)\end{matrix}$$

where $x$ is the year number and $r$ is the respondent number.

data {
 int n;
 int y[n];
 vector[n] x;
 int n\_s;
 int subj\_ind[n];
}
parameters {
 real beta0;
 real beta1;
 real z[n\_s];
 real<lower=0> tau;
}
transformed parameters {
 vector<lower=0,upper=1>[n] mu;
 for (i in 1:n) {
 mu[i] = inv\_logit(beta0 + beta1\*x[i] + z[subj\_ind[i]]);
 }
}
model {
 y ~ binomial(6, mu);
 z ~ normal(0, tau);
 target += -2\*log(tau);
}
generated quantities {
 vector[n] log\_lik;
 for (i in 1:n) {
 log\_lik[i] = binomial\_lpmf(y[i] | 6, mu[i]);
 }
}

saveRDS(BinomialRE, file = 'BinomialRE.Rds')

Model2Fit <- sampling(BinomialRE,
 list(n = nrow(d),
 x = d$Year,
 y = d$Response\_Numeric,
 n\_s = max(d$Respondent\_ID),
 subj\_ind = d$Respondent\_ID),
 iter = 10000,
 chains = mycores)

Model2Fit |> traceplot(pars = pars\_of\_interest)



summary(Model2Fit, pars = pars\_of\_interest)$summary |> kable(digits = 3)

|  | mean | se\_mean | sd | 2.5% | 25% | 50% | 75% | 97.5% | n\_eff | Rhat |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| beta1 | 0.182 | 0.001 | 0.08 | 0.025 | 0.128 | 0.18 | 0.235 | 0.34 | 13057.99 | 1 |

**1.8)** Consider again the rate at which the agreement with the viewpoint is increasing per year (i.e., $β\_{1}$), estimate the probability that this parameter differs between the models by more than 0.01 $\left(P\left[\left|β\_{1}^{Model2}−β\_{1}^{Model1}\right|>0.01\right]\right)$. Also give a short statement (1 sentence) about what the calculated probability implies (if anything). **[4]**

Model2Sims <- rstan::extract(Model2Fit)
cat('\n\n$P\\left[\\left|\\hat{\\beta\_1}^{Model2}-\\hat{\\beta\_1}^{Model1}\\right| > 0.01\\right]=$', round(mean(abs(Model2Sims$beta1 - Model1Sims$beta1) > 0.01), 3), '\n\n')

$P\left[\left|\hat{β\_{1}}^{Model2}−\hat{β\_{1}}^{Model1}\right|>0.01\right]=$ 0.952

**1.9)** Compare the two models using a criterion that considers model complexity and give a conclusion as to which model appears to offer a superior fit. Examples of acceptable criteria are LOOIC, DIC, and Bayes Factors, as well as variants of these. **[6]**

fits <- list(Normal = Model1Fit, Binomial = Model2Fit)

library(loo)
fits |> lapply(\(fit) {extract\_log\_lik(fit, merge\_chains = FALSE)}) -> log\_lik
log\_lik |> lapply(\(ll) {relative\_eff(exp(ll), cores = 1)}) -> r\_eff
fits |> length() |> seq\_len() |>
 lapply(\(i) {loo(log\_lik[[i]], r\_eff = r\_eff[[i]], cores = 1)}) |>
 loo\_compare() -> comparison
rownames(comparison) <- names(fits)[order(rownames(comparison))]
comparison |> knitr::kable(digits = 1)

|  | elpd\_diff | se\_diff | elpd\_loo | se\_elpd\_loo | p\_loo | se\_p\_loo | looic | se\_looic |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Binomial | 0.0 | 0.0 | -102.2 | 4.2 | 11.4 | 1.6 | 204.4 | 8.3 |
| Normal | -8.2 | 1.7 | -110.5 | 5.1 | 13.1 | 1.7 | 220.9 | 10.1 |

**1.10)** Using any of the models, what is the predicted standard deviation of a random future response in Year 6 of a random person who has not previously responded? **[4]**

# From Model 1:
nsims <- length(Model1Sims$beta1)
newRE <- rnorm(nsims, 0, Model1Sims$tau)
preds <- rnorm(nsims,
 Model1Sims$beta0 + Model1Sims$beta1\*6 + newRE,
 Model1Sims$sigma) |>
 round(digits = 0)
preds <- pmin(pmax(preds, 0), 6)
sd(preds)

| [1] 1.339049

# From Model 2:
nsims <- length(Model2Sims$beta1)
newRE <- rnorm(nsims, 0, Model2Sims$tau)
preds <- rbinom(nsims, 6,
 plogis(Model2Sims$beta0 + Model2Sims$beta1\*6 + newRE))
sd(preds)

| [1] 1.374577

**1.11)** Adapt the first mixed effects model to assume that the responses follow a conditional **ordered logistic** distribution given the year number as a continuous linear predictor and respondent as a random intercept on the logistic scale. This is also known as *ordinal regression*. Fit the model and summarise the distribution of the coefficient of the year number. **[5]**

Hint: It is critical that a strict prior be placed on the thresholds (e.g. N(0,10)). The key components of the model can be expressed mathematically as

$$\begin{matrix}y\_{i}&∼ordered logistic\left(μ\_{i},θ\right) i=1…n\\μ\_{i}&=β\_{0}+β\_{1}\*x\_{i}+z\_{r\_{i}}\\θ\_{1}&<θ\_{2}<θ\_{3}<θ\_{4}<θ\_{5} ∼truncN\left(0,10\right)\end{matrix}$$

where $x$ is the year number, $r$ is the respondent number, and $θ$ is the set of thresholds.

data {
 int n;
 int y[n];
 vector[n] x;
 int n\_s;
 int subj\_ind[n];
}
parameters {
 real beta0;
 real beta1;
 real z[n\_s];
 real<lower=0> tau;
 ordered[5] thresholds;
}
transformed parameters {
 vector[n] mu;
 for (i in 1:n) {
 mu[i] = beta0 + beta1\*x[i] + z[subj\_ind[i]];
 }
}
model {
 y ~ ordered\_logistic(mu, thresholds);
 z ~ normal(0, tau);
 target += -2\*log(tau);
 thresholds ~ normal(0, 10);
}
generated quantities {
 vector[n] log\_lik;
 for (i in 1:n) {
 log\_lik[i] = ordered\_logistic\_lpmf(y[i] | mu[i], thresholds);
 }
}

saveRDS(CatRE, file = 'CatRE.Rds')

Model3Fit <- sampling(CatRE,
 list(n = nrow(d),
 x = d$Year,
 y = d$Response\_Numeric,
 n\_s = max(d$Respondent\_ID),
 subj\_ind = d$Respondent\_ID),
 iter = 10000,
 chains = mycores,
 control = list(max\_treedepth = 12))

| Warning: There were 778 divergent transitions after warmup. See
| https://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
| to find out why this is a problem and how to eliminate them.

| Warning: There were 2528 transitions after warmup that exceeded the maximum treedepth. Increase max\_treedepth above 12. See
| https://mc-stan.org/misc/warnings.html#maximum-treedepth-exceeded

| Warning: There were 1 chains where the estimated Bayesian Fraction of Missing Information was low. See
| https://mc-stan.org/misc/warnings.html#bfmi-low

| Warning: Examine the pairs() plot to diagnose sampling problems

| Warning: The largest R-hat is 1.67, indicating chains have not mixed.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#r-hat

| Warning: Bulk Effective Samples Size (ESS) is too low, indicating posterior means and medians may be unreliable.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#bulk-ess

| Warning: Tail Effective Samples Size (ESS) is too low, indicating posterior variances and tail quantiles may be unreliable.
| Running the chains for more iterations may help. See
| https://mc-stan.org/misc/warnings.html#tail-ess

Model3Fit |> traceplot(pars = pars\_of\_interest)



summary(Model3Fit, pars = pars\_of\_interest)$summary |> kable(digits = 3)

|  | mean | se\_mean | sd | 2.5% | 25% | 50% | 75% | 97.5% | n\_eff | Rhat |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| beta1 | 0.394 | 0.014 | 0.179 | 0.052 | 0.272 | 0.395 | 0.517 | 0.741 | 173.58 | 1.02 |

**1.12)** Illustrate or estimate, and then analyse, the thresholds between response options as suggested by the ordinal regression. Are they evenly spaced (as the normal model assumes)? How do they relate to the thresholds implied by the binomial model?**[4]**

Model3Fit |> rstan::extract() -> postsims
boxplot(postsims$thresholds)



colMeans(postsims$thresholds)

| [1] -2.765421375 -1.275092379 0.006331686 1.463947283 3.945740897

qlogis((1:5)/6)

| [1] -1.6094379 -0.6931472 0.0000000 0.6931472 1.6094379

## Points total

The points on the test add up to **50**